# MAT 342 Applied Complex Analysis
# Solution to Midterm 2, April 17, 2013

**1.** (a) (10 points) Let $C$ be the positively oriented unit circle

$$z = e^{i\theta}, \quad -\pi \le \theta \le \pi,$$

and let $g(z) = z^{-1/2} = \exp\{-\frac{1}{2}\,\mathrm{Log}\ z\}$ be the principal branch of the power function $z^{-1/2}$. Using parametric representation for $C$, evaluate the integral

$$\int_C g(z)dz.$$

**Solution:** By the definition of the path integrals, we shall compute the following

$$\int_{-\pi}^{\pi} g(e^{i\theta})\frac{de^{i\theta}}{d\theta}d\theta = i\int_{-\pi}^{\pi} e^{-i\frac{\theta}{2}}e^{i\theta}d\theta = i\int_{-\pi}^{\pi} e^{i\frac{\theta}{2}}d\theta = 2e^{i\frac{\theta}{2}}|_{-\pi}^{\pi} = 4i$$

(b) (10 points) Let $C_r$ be a circle $|z| = r$, $0 < r < 1$, oriented counterclockwise, and suppose that $f(z)$ is analytic in the disk $|z| \le 1$. Show that there exists a constant $M > 0$ such that

$$\left|\int_{C_r} g(z)f(z)dz\right| \le 2\pi M\sqrt{r},$$

where $g(z)$ is the function from part (a).

**Solution:** We know that we have following inequality for *any* continuous function, with no requirement on its being analytic:

$$\left|\int_{C_r} h(z)dz\right| \le 2\pi r \sup_{C_r}|h(z)|$$

Since $f(z)$ is analytic -and hence continuous- we know that the quantity $M$ defined in the following way is a finite number:

$$M := \sup_{|z|\le 1} f(z)$$

In the case of the function $g(z)f(z)$ we have that

$$\sup_{C_r}|f(z)g(z)| \le \frac{M}{\sqrt{r}}$$

By combining these estimates we obtain that

$$\left|\int_{C_r} g(z)f(z)dz\right| \le 2\pi r\frac{M}{\sqrt{r}} = 2\pi M\sqrt{r}$$

**2.** Evaluate the following integrals

(a) (10 points)

$$\int_C \frac{e^{3z}}{z^3} dz,$$

where $C$ is the circle $|z-1| = 3$, oriented counterclockwise;

**Solution:** One might try a direct approach by parametrising the curve $C$, which leads to a difficult, if not impossible, integral if we are confined to the usual methods. Nevertheless, it's the magic of the Cauchy representation formula. Recall that for a general holomorphic function $f$ we have

$$f^{(n)}(0) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{\zeta^{n+1}} dz$$

wherein $C$ is any simple closed curve with the origin inside of it. If we take $f(z) = e^{3z}$, we shall have

$$\int_C \frac{e^{3z}}{z^3} dz = \frac{2\pi i}{2!} \frac{d^2 e^{3z}}{dz^2}|_{z=0} = 9\pi i$$

**Remark 0.1.** The function has to analytic for this formula to hold. Some students have taken the function $\frac{e^{3z}}{z^2}$, for which the Cauchy representation formula does not hold since it isn't analytic.

(b) (10 points)

$$\int_C \frac{dz}{z^5},$$

where $C$ is the boundary of a pentagon with vertices at $3i, \pm 3, \pm 2 - 2i$, oriented counterclockwise.

**Solution-1:** Note that the function $z^{-5}$ has a global primitive and therefore its integral along any closed path is zero.

**-2:** Another way of seeing this integral is by noting that the integrand is actually of the form $\frac{f(z)}{z^{4+1}}$, and therefore, is equal to the fourth derivative of the function $f(z) = 1$.

**-3:** Note that the integrand is analytic outside of the origin. One may therefore take any simple closed curve which can be deformed continuously to the pentagon $C$ as the path of integration. In particular, one may choose $\gamma = e^{it}$, the circle of radius 1 around the origin. This will lead to a simple integral that can be calculated.

**3.** (20 points) Let $C$ be any simple positively oriented closed contour, and let

$$f(z) = \int_C \frac{2\zeta^3 + 1}{(\zeta - z)^4} d\zeta.$$

Find $f(z)$ when $z$ is inside $C$, and when $z$ is outside $C$.

**Solution:** When $z$ is outside of the curve, the integrand is analytic since the denominator is everywhere non-zero. Therefore, by the Cauchy-Goursat theorem, we have that the integral is equal to 0.

When $z$ is in the interior, however, we use the same idea as in problem 2 part (b) to obtain

$$f(z) = \frac{2\pi i}{3!} \frac{d^3 \left(2\zeta^3 + 1\right)}{d\zeta^3} |_{\zeta = z} = 4\pi i$$

**Remark 0.2.** For the first part, when $z$ is ouside of the curve you have to justify why it is valid to use the Cauchy-Goursat theorem.

**4.** (20 points) Let $f(z) = u(x, y) + iv(x, y)$ be entire function, and suppose that the function

$$2u(x, y)v(x, y) = \mathrm{Im}[f(z)^2]$$

has an upper bound: there exists $c$ such that

$$u(x, y)v(x, y) \leq c \quad \text{for all} \quad (x, y) \in \mathbb{R}^2.$$

Show that $u(x, y)$ and $v(x, y)$ are constant functions.

**Solution:** Recall that $|e^z| = e^{\Re z}$ [1] Therefore, if we wish to solve the problem using composition with the exponential function and using Liouville's theorem about bounded holomorphic functions, we have to convert the information we have on the imaginary part of the function to the real part of another fuction.

Define $g(z) := e^{-if^2(z)}$. Since $\Re\left(-i\xi\right) = \Im\xi$, we have that $\Re\left(-if^2(z)\right) \leq c$. Hence, $|g(z)| \leq e^c$. Liouville's theorem tells us that $g(z)$ must be constant. Note that the exponential function is not invertible on $\mathbb{C}$, and, therefore, we have to prove that the constancy of $e^{f^2(z)}$ leads to constancy of $f^2(z)$. To see this, we may differentiate and show that the derivative $f'(z)$ vanishes and hence $f$ must be a constant function.

**Remark 0.3.** Of course the function $e^{f^2(z)}$ need not *a priori* be bounded if we only know about the imaginary part of $f^2(z)$.

**Remark 0.4.** A real valued function cannot be analytic since the Cauchy-Riemann condition cannot be satisfied for a real valued function. In particular, $u$ and $v$ cannot be analytic. For the same reason one can't talk about $u$ or $v$ being entire.

**5.** (20 points) Let $f(z) = \exp z^2$, and let $R$ be the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. Find the points in $R$ where $|f(z)|$ reaches its maximum and minimum values.

[1] $\Re$ and $\Im$ denote the real and imaginary parts of a complex number respectively.

**Solution:** Let $z = x + iy$. Then

$$e^{z^2} = e^{\Re z^2} = e^{x^2 - y^2}$$

and this tells us that the maximum and minimum occur at $(1,0)$ and $(0,1)$ respectively.

**Remark 0.5.** $|e^z| = e^{|z|}$ does not hold unless $z$ is a positive real number. Of course the inequality always holds.

**Remark 0.6.** Obtaining an upper bound for a function does not tell us about the point where the maxima or minima occur.

**Remark 0.7.** It's true that an the modulus of an analytic function obtains its maximum on the boundary, nevertheless, checking the *corner points* will not in general suffice. Boundary of the rectangle is all its sides, not just the vertices.